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A scheme for the electrochemical machining of metals by a cathode tool with a curvilinear part of the boundary☆

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ABSTRACT

The solution of a non-linear plane problem in the theory of the electrochemical machining of metals, associated with the determination of the shape of a surface (the anode) during its treatment with a cathode tool with a curvilinear part of the boundary, is obtained by methods developed for problems of jet flow past curvilinear obstacles. A condition is obtained which is identical to the well known Brillouin-Villat condition in fluid dynamics for smooth separation, the use of which enables one to determine the position of the transition point from the zone of steady treatment conditions into a region where the dissolution of the metal does not occur. The fixed shapes of the anode boundary are found for two cathode configurations.

Formulations of problems within the limits of the model of an ideal process for the electrochemical machining of metals and methods for solving them have been presented in Refs. [1,2]. The magnitude of the current efficiency for anode metal solution reactions, which takes account of processes occurring on the anode surface accompanying the solution of the metal and is equal to the fraction of the charge consumed solely in dissolving the metal, is an important parameter of the process. The hodograph method [2] enables the two-dimensional fixed anode boundary to be determined, assuming the current efficiency to be constant, in problems with a given geometry of a cathode with straight boundaries. Numerical methods are used to solve the problem of electrochemical machining by a cathode with curvilinear parts of the boundary and a review of these is available [1]. The two-dimensional problem of the determining the fixed boundary of the anode, taking account of the dependence of the current efficiency on the anode current density has been solved [3] using the boundary-element method.

A numerical-analytical solution of this problem in the case of a cathode of symmetrical shape with a curvilinear part of the boundary is obtained below within the limits of the model of an ideal process.[1] Methods developed for solving problems on jet flow past curvilinear obstacles [4–6] and problems in the theory of seepage and explosions [7,8] are investigated. The transition from a zone of intense solution of the metal into a region where the anode current is very small and solution of the metal barely occurs, which is characteristic of electrolytes that are solutions of sodium nitrate and sodium chlorate, is taken into account.

1. Model of the process

According to the model of the process described earlier,[3] an electric field with a potential u in the inter-electrode gap is described by Laplaces equation

$$\nabla^2 u = 0. \tag{1.1}$$

The values of the potentials u_a and u_c on the surfaces of the anode and cathode are constant.

In the case of solutions of sodium nitrate and sodium chlorate, the dependence of the current efficiency on the anode current density i_a can be represented in the form [3]

$$\eta(i_a) = \begin{cases} 0, & i_a \le i_{\rm cr} \\ a_0 + a_1/i_a, & i_a > i_{\rm cr} \end{cases}; \quad a_0, a_1, i_{\rm cr} \text{ — are constants} \tag{1.2}$$

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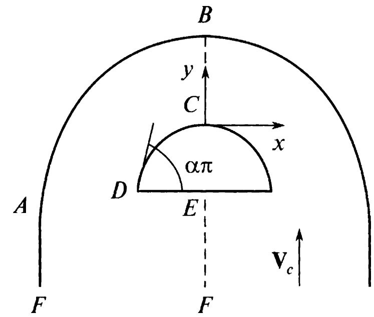

Fig. 1.

The distribution of the normal derivative of the potential on the fixed anode boundary has the form [3]

$$\kappa\frac{\partial u}{\partial n_a} = -\frac{a_1}{a_0} + \frac{\rho V_c}{a_0\,\varepsilon}\cos\theta \tag{1.3}$$

where κ is the electrical conductivity of the medium, ε is the electrochemical equivalent of the metal, ρ is the density of the anode material, and θ is the angle between the velocity vector $\mathbf{V}_c$ of the cathode feed and the vector $\mathbf{n}_a$ of the external normal at a given point of the anode boundary.

A plane problem is considered and a system of Cartesian coordinates x_1, y_1, associated with the cathode, is introduced. It is assumed that the cathode moves in the direction of the ordinate.

Starting from Eq. (1.1), the function $u(x_1,y_1)$ can be assumed to be the imaginary part of the analytic function $f(z_1) = \upsilon(x_1,y_1) + iu(x_1,y_1)$ of the complex variable $z_1 = x_1 + iy_1$. The function $f(z_1)$ is the complex potential of the electrostatic field and its real part $\upsilon(x_1,y_1)$ is a function of the current. The contour lines $u(x_1,y_1)$ are equipotential field lines and the lines $\upsilon(x_1,y_1) = \text{const}$ are force lines.[9]

We now introduce a characteristic current density i_0 in the treatment which characterizes the length H[2] and the dimensionless variables

$$i_0 = \rho V_c/\varepsilon, \quad H = \kappa(u_a - u_c)/i_0, \quad x = x_1/H, \quad y = y_1/H, \quad n = n_a/H$$

and change to the dimensionless complex potential

$$W(z) = \varphi(x,y) + i\psi(x,y); \quad z = x + iy$$

using the transformation[2]

$$W(z) = (f(z) - iu_c)/(u_a - u_c)$$

The function ψ then satisfies Laplace's equation in the inter-electrode gap with the following conditions on the electrode boundaries

$$\psi_a = 1, \quad \psi_c = 0 \tag{1.4}$$

and, on the unknown anode boundary

$$\partial\psi/\partial n = a + b\cos\theta; \quad a = -a_1/(a_0 i_0), \quad b = 1/a_0 \tag{1.5}$$

In special cases, the cathode can have lines of symmetry or segments of the boundaries made of dielectric coatings on which the condition

$$\partial\psi/\partial n = 0 \tag{1.6}$$

is satisfied.

The plane potential electric field is simulated by a fictitious plane-parallel potential flow of an ideal incompressible fluid and the stream function is specified in conformity with the electric field potential. The hydrodynamic analogue of the electric field strength **E** is the flow velocity **V**, and the vectors **V**, and **E** are mutually orthogonal.[9] According to condition (1.6), the velocity of the fictitious flow on the anode boundary varies as follows:

$$V = a + b\cos\theta \tag{1.7}$$

where θ is the argument of the velocity vector.

2. Formulation of the problem and its numerical-analytical solution

A diagram of a section of the inter-electrode gap for the problem considered is presented in Fig. 1. The cathode boundary is a symmetric contour with a curvilinear part. By virtue of the symmetry of the inter-electrode gap, we will restrict ourselves to considering of its left-hand part. In it, the line *CDE* corresponds to the boundary of the cathode, and the lines of symmetry *BC* and *EF* are electric current lines which are orthogonal to the equipotential field lines. The vector V_c indicates the direction of feed of the cathode. The abscissa is chosen to be orthogonal to the direction of feed of the cathode. The angles the tangent makes with the arc *CD* at the points *C* and *D* and the abscissa are equal to zero and $\alpha\pi$ respectively.

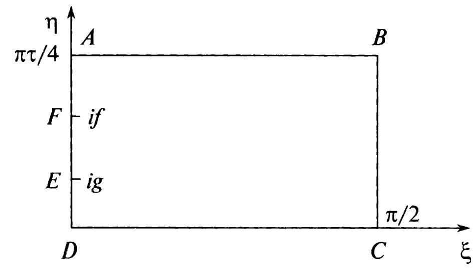

Fig. 2.

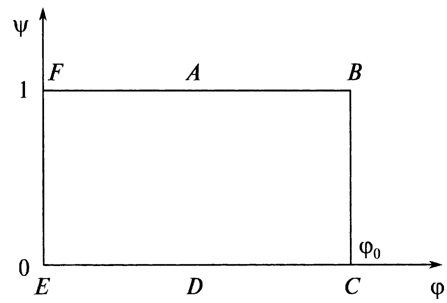

Fig. 3.

We divide the required anode boundary into two regions. Solution of the metal occurs in the region AB. The normal derivative $\partial\psi/\partial n$ in this segment satisfies condition (1.6). In the region which is simulated by the vertical straight segment AF, the anode current efficiency is practically equal to zero and solution of the metal does not occur. The current density in the segment AF varies from the value i_{cr} at point A to zero at the infinitely remote point F.

The problem of the theory of plane steady flows of an ideal incompressible fluid, according to the definition of the boundary AB with a given law of variation in the velocity (1.8), is the hydrodynamic analogy. The flow is created by a system of continuously distributed sources along the line EF and sinks on the line BC.

To solve the problem, we will introduce the auxiliary complex variable $t=\xi+i\eta$, which varies in the region $D_t(0\le\xi\le\pi/2, 0\le\eta\le h)$, where $h=\pi|\tau|/4$, $\tau=i|\tau|$ (Fig. 2) and we will seek the function $z(t)$, which conformally maps the rectangle D_t into the flow domain with the correspondence of points shown in Figs. 1 and 2.

Instead of the function $z(t)$, the Zhukovskii function [5]

$$\chi(t)=\ln\left(\frac{1}{V_0}\frac{dW}{dz}\right)=r-i\theta;\quad r=\ln\frac{V}{V_0} \tag{2.1}$$

can be sought, where $V_0=a+b$ is the value of the velocity of the fictitious flow at the point $B(t=\pi/2+ih)$. The function $\chi(t)$ is connected with the functions $W(t)$ and $z(t)$ by the relation

$$\frac{dz(t)}{dt}=\frac{\exp(-\chi(t))}{V_0}\frac{dW}{dt} \tag{2.2}$$

According to conditions (1.5), the complex potential $W(t)=\varphi(t)+i\psi(t)$ satisfies the following boundary conditions

$$\psi(t)=\begin{cases}0, & t=\xi,\quad \xi\in[0,\pi/2],\quad t=i\eta,\quad \eta\in[0,g]\\ 1, & t=i\eta,\quad \eta\in[f,h],\quad t=\xi+ih,\quad \xi\in[0,\pi/2]\end{cases}$$

It follows from condition (1.7) that the function $\varphi(t)$ takes constant values on the lines of symmetry DF and BC. Without loss of generality, we shall assume that

$$\varphi(t)=\begin{cases}0, & t=i\eta,\quad \eta\in[g,f]\\ \varphi_0, & t=\pi/2+i\eta,\quad \eta\in[0,h]\end{cases}$$

The region of variation of the complex potential is a rectangle with sides φ_0 and 1 (Fig. 3). Using the method of conformal mappings, we find the derivative of the complex potential

$$\frac{dW}{dt}=N_1F_1(t),\quad N_1=\left[\int\limits_g^f F_1(ix)dx\right]^{-1},\quad F_1(t)=\sqrt{\frac{R_1(t)}{R_2(t)}} \tag{2.3}$$

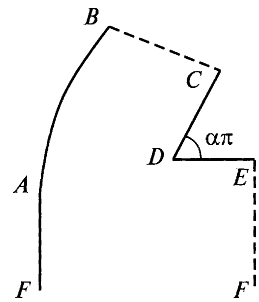

Fig. 4.

where

$$R_1(t) = (\vartheta_3(2t)\vartheta_2(0) - \vartheta_2(2t)\vartheta_3(0))(\vartheta_3(2t)\vartheta_3(0) - \vartheta_2(2t)\vartheta_2(0))$$

$$R_2(t) = (\vartheta_2(2gi)\vartheta_3(2t) - \vartheta_3(2gi)\vartheta_2(2t))(\vartheta_2(2fi)\vartheta_3(2t) - \vartheta_3(2fi)\vartheta_2(2t))$$

Henceforth, $\vartheta_i(u)(i = 1,2,3,4)$ are theta-functions for the periods π and $\pi\tau$.[10]

The parameter φ_0, which characterizes the current in the electrochemical cell,[1] is defined by the formula

$$\varphi_0 = N_1 \int_0^{\pi/2} F_1(x)dx$$

We will now consider the boundary conditions for the function $\chi(t)$. On the polygonal segments of the boundary, its imaginary part is a piecewise-constant function

$$\operatorname{Im}\chi(t) = \begin{cases} -\pi, & t = i\eta, \quad \eta \in [0, f) \\ -\pi/2, & t = i\eta, \quad \eta \in (f, h] \\ 0, & t = \pi/2 + i\eta, \quad \eta \in [0, h] \end{cases} \tag{2.4}$$

Suppose a continuous function $\theta(s)$, where s is the length of an arc measured from point D (Fig. 1), is given for the arc CD. The expression

$$K(s(\theta)) = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\theta}{d\xi}\frac{d\xi}{ds}\right|$$

where $d\xi/ds = dt/dz$, holds for the curvature of the arc CD. According to the scheme (Fig. 1), the conditions $d\theta/d\xi < 0$ and $d\xi/ds > 0$ hold on the arc CD and, consequently,

$$\frac{d\theta}{d\xi} = -K(\theta)\frac{ds}{d\xi} = -K(\theta)\left|\frac{dz}{dt}\right|$$

From this, when account is taken of equality (2.2), we have

$$\frac{d\theta}{d\xi} = -\frac{K(\theta)}{V_0}\left|\frac{dW}{d\xi}\right|\exp(-r(\xi)), \quad \xi \in [0, \pi/2] \tag{2.5}$$

Using equality (2.1) and condition (1.8), we obtain the condition on the unknown anode boundary AB

$$a + b\cos\theta(t) - V_0\exp(r(t)) = 0, \quad t = \xi + ih, \quad \xi \in [0, \pi/2] \tag{2.6}$$

$$r(\pi/2 + ih) = 0 \tag{2.7}$$

We will represent the function $\chi(t)$ in the form of the sum:[5]

$$\chi(t) = \chi_*(t) + \Omega_1(t) + \Omega_2(t) \tag{2.8}$$

where $\Omega_k(t) = v_k(t) + i\varepsilon_k(t)(k = 1,2)$ are analytic over the range of variation of the variable t of the functions. The Zhukovskii function $\chi_*(t) = r_*(t) - i\theta_*(t)$, where $r_* = \ln(V_*/V_0)$, corresponds to the flow diagram (Fig. 4) in which the curvilinear arc DC is replaced by the segment of a line $\theta_*(\xi) = \theta(0), \xi \in [0, \pi/2]$ and the equality $V_* = V_0$ is satisfied on the boundary AB, that is, $\operatorname{Re}\chi_*(\xi + ih) = 0, \xi \in [0, \pi/2]$. The functions $\chi(t)$ and $\chi_*(t)$ have the same singularities in the range of variation of the variable t.

Using Chaplygin's method of singular points, we obtain

$$\chi_*(t) = \frac{1}{2}\ln\frac{\vartheta_1(t - if)\vartheta_1(t + if)}{\vartheta_4(t - if)\vartheta_4(t + if)} - 2(1 - \alpha)\ln\frac{\vartheta_1(t)}{\vartheta_4(t)} - \alpha\pi i \tag{2.9}$$

We require the following boundary conditions for the unknown functions $\Omega_k(t)(k=1,2)$ to be satisfied

$$\begin{aligned}&\varepsilon_1(t)=\varepsilon_2(t)=0,t=i\eta,\eta\in[0,h];\quad \varepsilon_2(t)=0,t=\xi,\xi\in[0,\pi/2]\\&\varepsilon_1(t)=\alpha\pi,\varepsilon_2(t)=0,t=\pi/2+i\eta,\eta\in[0,h]\\&\nu_1(t)=0,t=\xi+ih,\xi\in[0,\pi/2]\end{aligned} \tag{2.10}$$

It then follows from relations (2.5), (2.8) and (2.9) that

$$\frac{d\varepsilon_1}{d\xi}=\frac{K(\theta(\xi))}{V_0}\rho(\xi)\exp(-(\nu_1(\xi)+\nu_2(\xi))),\quad \xi\in[0,\pi/2] \tag{2.11}$$

where

$$\rho(\xi)=\left|\frac{dW}{d\xi}\right|\exp(-r_*(\xi))=N_1F_2(\xi)$$

$$F_2(\xi)=F_1(\xi)\sqrt{\frac{\vartheta_4(\xi-if)\vartheta_4(\xi+if)}{\vartheta_1(\xi-if)\vartheta_1(\xi+if)}}\left(\frac{\vartheta_1(\xi)}{\vartheta_4(\xi)}\right)^{2(1-\alpha)}$$

Using conditions (2.6) and (2.7), we obtain

$$a+b\cos\theta(t)-V_0\exp\nu_2(t)=0,\quad \nu_2(\pi/2+ih)=0 \tag{2.12}$$

where

$$\theta(t)=\theta_*(t)-\varepsilon_1(t)-\varepsilon_2(t),\quad t=\xi+ih,\quad \xi\in[0,\pi/2]$$

Hence, for the unknown functions $\Omega_k(t)(k=1,2)$, we have the boundary value problem defined by boundary conditions (2.10)–(2.12).

By virtue of conditions (2.10), these functions are expanded in series with real coefficients (everywhere, summation is subsequently carried out from $n=1$ to $n=\infty$)

$$\Omega_1(t)=2\alpha(h+it)+2\sum c_n\,sh2(h+it)n \tag{2.13}$$

$$\Omega_2(t)=b_0+\sum b_n\cos 2tn,\quad b_0=\sum(-1)^{n+1}b_n\operatorname{ch}2hn \tag{2.14}$$

When account is taken of expansions (2.13) and (2.14), condition (2.11) can be reduced to the form

$$2\alpha+4\sum c_n n\cos 2\xi n\operatorname{ch}2hn=N_2K(\theta(\xi))\frac{F_2(\xi)}{F_3(\xi)},\quad \xi\in[0,\pi/2] \tag{2.15}$$

where

$$N_2=\frac{N_1}{V_0\exp(b_0+2\alpha h)},\quad \theta(\xi)=\alpha(\pi-2\xi)-2\sum c_n\sin 2\xi n\operatorname{ch}2hn$$

$$F_3(\xi)=\exp\left(2\sum c_n\cos 2\xi n\operatorname{sh}2hn+\sum b_n\cos 2\xi n\right)$$

Integrating expression (2.15) with respect to the variable ξ in the interval $[0,\pi/2]$, we obtain

$$N_2\int_0^{\pi/2}K(\theta(\xi))\frac{F_2(\xi)}{F_3(\xi)}d\xi=\alpha\pi \tag{2.16}$$

We will represent condition (2.12) in the form

$$a+b\cos T(\xi)-V_0\exp\omega(\xi)=0,\quad \xi\in[0,\pi/2] \tag{2.17}$$

where

$$T(\xi)=\theta(\xi+ih)=T_*(\xi)-\mu_1(\xi)-\mu_2(\xi)$$

$$\omega(\xi)=\nu_2(\xi+ih)=b_0+\sum b_n\cos 2\xi n\operatorname{ch}2hn$$

$$T_*(\xi)=\theta_*(\xi+ih)=\frac{1}{2}\arg\frac{\vartheta_4(\xi+i\gamma_1)\vartheta_4(\xi+i\gamma_2)}{\vartheta_1(\xi+i\gamma_1)\vartheta_1(\xi+i\gamma_2)}$$

$$+2(1-\alpha)\arg\frac{\vartheta_1(\xi+ih)}{\vartheta_4(\xi+ih)}+\alpha\pi,\quad \gamma_1=h-f,\quad \gamma_2=h+f$$

$$\mu_1(\xi)=\varepsilon_1(\xi+ih)=2\alpha\xi+2\sum c_n\sin 2\xi n,\quad \mu_2(\xi)=\varepsilon_2(\xi+ih)=-\sum b_n\sin 2\xi n\operatorname{sh}2hn$$

Differentiating expression (2.17) with respect to the variable ξ, we conclude that

$$T'(\xi) = 0 \text{ when } \xi = 0 \tag{2.18}$$

This condition is identical to the well known Brillouin-Villat condition [5,6] in fluid dynamics for smooth separation. It can be represented in the form

$$T_*'(0) - 2\alpha - 4\sum c_n n + 2\sum b_n n \,\mathrm{sh}\, 2hn = 0 \tag{2.19}$$

where

$$T_*'(0) = \frac{1}{2}\mathrm{Im}\left(\frac{\vartheta_4'(i\gamma_1)}{\vartheta_4(i\gamma_1)} + \frac{\vartheta_4'(i\gamma_2)}{\vartheta_4(i\gamma_2)} - \frac{\vartheta_1'(i\gamma_1)}{\vartheta_1(i\gamma_1)} - \frac{\vartheta_1'(i\gamma_2)}{\vartheta_1(i\gamma_2)}\right) + 2(1-\alpha)\mathrm{Im}\left(\frac{\vartheta_1'(ih)}{\vartheta_1(ih)} - \frac{\vartheta_4'(ih)}{\vartheta_4(ih)}\right)$$

All the necessary geometrical flow characteristics can be found using parametric relation (2.2):

$$dz = \frac{N_1 \exp(\alpha\pi i)}{V_0} \frac{F_2(t)}{\exp(\Omega_1(t) + \Omega_2(t))} dt \tag{2.20}$$

Integrating equality (2.20) in the segment $[0, ig]$, we find the length of the segment DE

$$L = L(f, g, \tau) \tag{2.21}$$

3. A special case

As an example, we will consider the case when the curvilinear part of the cathode boundary is an arc of an ellipse, the foci of which are located on the ordinate. The curvature of the ellipse is described in the form

$$K(\theta) = \left(1 - \varepsilon^2 \sin^2\theta\right)^{3/2} \Big/ p, \quad p = a_2^2/b_2, \quad \varepsilon = \sqrt{b_2^2 - a_2^2}\Big/b_2 \tag{3.1}$$

where a_2 and b_2 are the semi-axes of the ellipse. Substituting expression (3.1) into formula (2.15), after integration with respect to the variable ξ in the segment $[0, \pi/2]$, we obtain

$$p = \frac{N_2}{\alpha\pi} \int_0^{\pi/2} \left(1 - \varepsilon^2 \sin^2\theta(\xi)\right)^{3/2} \frac{F_2(\xi)}{F_3(\xi)} d\xi \tag{3.2}$$

For given values a_2 and b_2 of the semi-axes of the ellipse, the length L, the angle $\alpha\pi$, the coefficients a_0 and a_1, characterizing the properties of the electrolyte and the typical current density i_0, the coefficients of the expansions (2.13) and (2.14) and the parameters g, f and $|\tau|$ are determined from Eqs (2.15), (2.17), (2.19), (2.21) and (3.2). The coefficients of the expansions (2.13) and (2.14) are determined in such a way that condition (2.15) is satisfied on the boundary of the cathode and condition (2.17) is satisfied on the unknown boundary of the anode.

The problem is solved numerically by the collocation method. The system of equations for calculating the coefficients of expansions (2.13) and (2.14) is solved by Newton's method together with Eqs (2.19), (2.21) and (3.2), which are intended for determining the parameters g, f and τ. Then, using the parametric relation (2.20), it is possible to find all the necessary geometrical characteristics including the coordinates of the points of the anode and cathode boundaries.

In the special case when $a_2 = b_2 = R$, these formulae give the solution of the problem when the curvilinear part of the boundary of the cathode is an arc of a circle of radius R which, according to equality (3.2), is expressed using the formula

$$R = \frac{N_2}{\alpha\pi} \int_0^{\pi/2} \frac{F_2(\xi)}{F_3(\xi)} d\xi \tag{3.3}$$

4. Results of the calculations

When

$$a_2 = 0.15, \quad b_2 = 0.25, \quad L = 0.15, \quad \alpha = 0.5,$$

$$i_0 = 50\ \mathrm{A/cm^2}, \quad a_0 = 0.906, \quad a_1 = -12.82$$

the results of the calculation of the magnitude of the gap L_1 in the section BC and the coordinates of the point A are: $L_1 = 0.426$, $x = -0.979$, $y = -1.046$ (see the solid lines in Fig. 5). For the special case $a_2 = b_2 = R = 0.25$ when $L = 0.25$ and all the values of the remaining parameters are the same as in the preceding case, the results are: $L_1 = 0.475$, $x = -1.105$, $y = -1.147$ (see the dashed lines in Fig. 5).

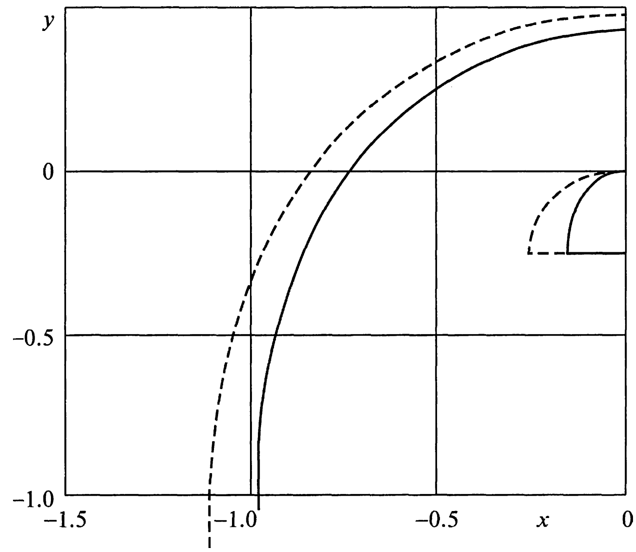

Fig. 5.

References

1. Davydov AD, Kozak Ye. *High-Speed Electrochemical Shaping*. Moscow: Nauka; 1990.
2. Karimov A Kh, Klokov VV, Filatov YeI. *Methods of calculating Electrochemical Shaping*. Kazan: Izd Kazan Univ; 1990.
3. Kotlyar LM, Minazetdinov NM. Determination of the shape of an anode taking into account the properties of the electrolyte in problems of the electrochemical dimensional machining of metals. *Zh Prikl Mekh Tekh Fiz* 2003;**44**(3):179–84.
4. Bikhoff G, Zarantonello E. *Jets, Wakes and Cavities*. New York: Acad. Press; 1957.
5. Gurevich MI. *Theory of Jets in Ideal Fluids*. New York: Academic; 1965.
6. Kotlyar LM. The case of jet flow in a heavy fluid. *Izv Akad Nauk SSSR MZhG* 1975;**3**:140–3.
7. Kotlyar L, Skvortsov EV. Seepage of a viscoplastic fluid to a sink in a curvilinear stratum. *Dokl Akad Nauk SSSR* 1972;**209**(5):1049–52.
8. Kotlyar LM. The explosion on the surface of the ground of a linearly distributed charge of curvilinear shape. *Zh Prikl Mekh Tekh Fiz* 1975;**1**:187–91.
9. Lavrent'ev MA, Shabat BV. *Methods of the Theory of Functions of a Complex Variable*. Moscow: Nauka; 1987.
10. Whitaker ET, Watson GN. *A Course of Modern Analysis*. Cambridge: Cambridge University Press; 1927.

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